ECON 210: Section 9

Marcelo Sena

Plan for today

- Portfolio Problems
  - portfolio choice with CARA utility
  - useful class of models with analytical tractability
- Previous years exams

Finite Horizon with CARA Utility

- Agent derives utility from consumption over T periods, discounting at rate β.
- Utility is

$$u(c_t, t) = -\frac{\exp(-\alpha c_t)}{\alpha},\ t \in \{1, \ldots, T\} \tag{1}$$

- There is a risk-free with return R^f and risky asset with normally distributed returns with mean μ and variance σ.
- Show that the value function takes the form

$$V(t, W_t) = -\frac{\delta(t)}{\beta} \frac{e^{-\alpha(t) W_t}}{\alpha(t)} \tag{2}$$

- What is the optimal portfolio and consumption choice in the limit $T \to \infty$? Provide comparative statics with respect to R^f, μ, σ.

Finite Horizon with CARA Utility

Solution

Start by writing the budget constraint. Let $A_{f,t}$ and $A_{r,t}$ be the dollar amount invested in the risk-free asset and risky asset. Next period wealth is given by:

$$W_{t+1} = R^f A_{f,t} + R_{t+1} A_{r,t} \tag{3}$$

The resource constraint of the agent is $W_t = A_{f,t} + A_{r,t} - c_t$, from which we find that the law of motion of wealth can be written as

$$W_{t+1} = R^f (W_t - A_{r,t} - c_t) + R_{t+1} A_{r,t} = R^f (W_t - c_t) + A_{r,t}(R_{t+1} - R^f) \tag{4}$$

Finite Horizon with CARA Utility

Solution

The optimization problem is

$$\max_{c_t, A_{r,t}} \sum_{t=0}^{T} -\frac{\exp(-\alpha c_t)}{\alpha} \tag{5}$$

$$\text{s.t. } W_{t+1} = R^f(W_t - c_t) + A_{r,t}(R_{t+1} - R^f) \tag{6}$$

The recursive problem is

$$V(t, W_t) = \max_{c_t, A_{r,t}} -\frac{\exp(-\alpha c_t)}{\alpha} + \beta E_t \left[V(t+1, W_{t+1})\right] \tag{7}$$

$$\text{s.t. } W_{t+1} = R^f(W_t - c_t) + A_{r,t}(R_{t+1} - R^f) \tag{8}$$

Finite Horizon with CARA Utility

Solution

Key step for analytical tractability:

- calculate the integral in the continuation value
- with exponential utility and log-normality
  - $\implies$ tractable integral

In continuous time we have more tractability

- we can express expected continuation values as derivatives!
- also easier to compute
  - integrals are computationally intensive

Finite Horizon with CARA Utility

Solution

From our guess of the value function, we can calculate the expectation term

$$E_t[V(t+1, W_{t+1})] = E_t\left[-\frac{\delta(t+1)}{\beta}\frac{\exp\left(-\alpha(t+1)\left(R^f(W_t-c_t)+A_{r,t}(R_{t+1}-R^f)\right)\right)}{\alpha(t+1)}\right] \quad (9)$$

$$= -\frac{\delta(t+1)}{\beta}\frac{E_t\left[\exp\left(-\alpha(t+1)\left(R^f(W_t-c_t)+A_{r,t}(R_{t+1}-R^f)\right)\right)\right]}{\alpha(t+1)} \quad (10)$$

$$= -\frac{\delta(t+1)}{\beta}\frac{\exp\left(-\alpha(t+1)\left(R^f(W_t-c_t)+A_{r,t}(\mu-R^f)\right)+\frac{1}{2}\alpha(t+1)^2A_{r,t}^2\sigma^2\right)}{\alpha(t+1)} \quad (11)$$

$$(12)$$

where the last equality calculates uses the mean of a log-normal random variable.
Plugging back into the Bellman equation and taking first order conditions with respect to c_t and $A_{r,t}$ yields:

$$\alpha(t+1)(\mu-R_f)-\alpha^2(t+1)\sigma^2A^*_{r,t}=0 \implies A^*_{r,t}=\underbrace{\frac{\mu-R_f}{\sigma}}_{:=\text{sharpe ratio}}\frac{1}{\sigma\alpha(t+1)} \quad (13)$$

$$\exp\left(-\alpha c_t^*\right)-\delta(t+1)R_f\exp\left(-\alpha(t+1)\left[\left(W_t-c_t^*\right)R_f+A_{r,t}\left(\mu-R_f\right)\right]+\frac{1}{2}\alpha^2(t+1)A^{*2}_{r,t}\sigma^2\right)=0 \quad (14)$$

Finite Horizon with CARA Utility

Solution

Plugging optimal policies back into the Bellman equation and after some algebra, we can show that

$$V(t, W_t) = k(t+1) \exp^{-\frac{R_f W_t \alpha \alpha(t+1)}{R_f \alpha(t+1)+\alpha}} \tag{15}$$

where importantly $k(t+1)$ does not depend on W_t

$$k(t) = -\alpha\delta(t+1) e^{-\frac{0.5R_f^2}{\sigma^2}} e^{-\frac{0.5\mu^2}{\sigma^2}} e^{\frac{1.0R_f\mu}{\sigma^2}} e^{\frac{0.5R_f^3\alpha(t+1)}{R_f\sigma^2\alpha(t+1)+\alpha\sigma^2}} e^{\frac{0.5R_f\mu^2\alpha(t+1)}{R_f\sigma^2\alpha(t+1)+\alpha\sigma^2}} \cdot$$
$$e^{-\frac{R_f^2\mu\alpha(t+1)}{R_f\sigma^2\alpha(t+1)+\alpha\sigma^2}} e^{-\frac{R_f\sigma^2\alpha(t+1)\log(R_f)}{R_f\sigma^2\alpha(t+1)+\alpha\sigma^2}} e^{-\frac{R_f\sigma^2\alpha(t+1)\log(\delta(t+1))}{R_f\sigma^2\alpha(t+1)+\alpha\sigma^2}} -$$
$$\alpha(t+1) e^{-\frac{0.5R_f^2\alpha}{R_f\sigma^2\alpha(t+1)+\alpha\sigma^2}} e^{-\frac{0.5\alpha\mu^2}{R_f\sigma^2\alpha(t+1)+\alpha\sigma^2}} e^{\frac{R_f\alpha\mu}{R_f\sigma^2\alpha(t+1)+\alpha\sigma^2}} e^{\frac{\alpha\sigma^2\log(R_f)}{R_f\sigma^2\alpha(t+1)+\alpha\sigma^2}} e^{\frac{\alpha\sigma^2\log(\delta(t+1))}{R_f\sigma^2\alpha(t+1)+\alpha\sigma^2}} \tag{16}$$

Finite Horizon with CARA Utility

Solution

Note then that for out conjecture to be true, we must have:

$$-\frac{\delta(t)}{\beta}\frac{e^{-\alpha(t)W_t}}{\alpha(t)} = k(t+1)\exp^{-\frac{R_f W_t \alpha\alpha(t+1)}{R_f\alpha(t+1)+\alpha}} \tag{17}$$

$$\implies \alpha(t) = \frac{R_f\alpha\alpha(t+1)}{R_f\alpha\left(t+1\right)+\alpha} \tag{18}$$

Equation (18) is a first-order difference equation. Solving it yields a necessary condition for our guess to be correct. We need one boundary condition to solve this difference equation, which is given by the last period condition:

$$\alpha(T) = \alpha \tag{19}$$

It remains then to determine $\delta(t)$, which can be done through an analogous procedure.

Finite Horizon with CARA Utility

Solution

- similar computations also arise in pricing assets with exponentially-affine stochastic discount factors (Duffie and Kan (1996))
- useful class of models to connect asset-prices to economic fundamentals state variables

Duffie, Darrell and Rui Kan, "A yield-factor model of interest rates," *Mathematical finance*, 1996, *6* (4), 379–406.