ECON 210: Section 5

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Plan for today

- Dynamic Contracts with Dynamic Programming
  - an example of "dynamic programming squared"
- Questions and open discussion

Dynamic Contracts with Dynamic Programming

- study a version of Albuquerque and Hopenhayn (2004)
  - theory of endogenous borrowing constraints
  - deterministic version of the model

Dynamic Contracts with Dynamic Programming

Setup

- entrepreneur may start a firm that requires setup cost of I
- if project is started and capital k_t is installed, project generates cash-flow $\pi(k_t)$

$$\pi(k_t) = R(k_t) - (1 + r)k_t \tag{1}$$

- assume unique value (call it k^*) such that π is maximized

Dynamic Contracts with Dynamic Programming

Frictions

- project requires external financing from a lender
- moral hazard: entrepreneur can runaway with capital and default on lender
- $\implies$ lending contract requires

$$\text{value of staying in contract } > \text{running away} \tag{2}$$

Contract

Focus on a set of contracts $C \in \mathcal{C}$ that specify

- initial loan I
- capital k_t (think of this as working capital, i.e., requires financing)
- cash-flow for entrepreneur and lender

Contracts can be contingent on the full history of all variables

Preferences and Technology

Given a contract $C = \{I, k_t, d_t, \tau_t,\}_t$

$$U_e(C) = \sum_{t=1}^{\infty} \frac{d_t}{(1+r)^t} \tag{3}$$

$$U_l(C) = \sum_{t=1}^{\infty} \frac{\tau_t}{(1+r)^t} \tag{4}$$

Define also the value of the firm

$$W(C) = \sum_{t=1}^{\infty} \frac{\pi(k_t)}{(1+r)^t} \tag{5}$$

timing: at $t = 0$, lender offers a take-it-or-leave-it contract C

Contract with perfect enforceability

$$\max_{k_t \geq 0, d_t \geq 0, \tau_t \geq 0} U_I \tag{6}$$

$$s.t. \begin{cases} U_e \geq \bar{U} \\ \tau_t + d_t = \pi(k_t) \end{cases} \tag{7}$$

What is the optimal contract in this case?

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- $k_t = k^*$

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What is the optimal contract in this case?

- choose d_t so that $U_e \geq \bar{U}$ binds
- $k_t = k^*$
- as we vary $\bar{U}$, construct contracts with same allocation but different dividend d_t and debt repayment τ_t
  - a manifestation of Modigliani and Miller (1958)

Contract with limited enforceability

We assume

$$I < \frac{\pi(k^*)}{r} < I + k^* \tag{8}$$

What happens if the lender decides to start the project at full scale?

- value to entrepreneur $= W - I = \frac{\pi(k^*)}{r} - I < k^*$

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- value to entrepreneur $= W - I = \frac{\pi(k^*)}{r} - I < k^*$
- what is the interpretation of the assumption above?
- assumption implies that even though the firm is worth the investment, its not possible to start at full scale

Contract with limited enforceability

$$\max_{k_t \geq 0, d_t \geq 0, \tau_t} U_l \tag{9}$$

$$s.t. \begin{cases} U_e \geq \bar{U} \\ \tau_t + d_t = \pi(k_t) \\ \sum_{s=t}^{\infty} \frac{d_s}{(1+r)^s} \geq k_t \forall t \end{cases} \tag{10}$$

Maximize instead value of the firm (equivalent problem, why?)

$$\max_{k_t \geq 0, d_t \geq 0} W_0 \tag{11}$$

$$s.t. \begin{cases} U_e \geq \bar{U} \\ \sum_{s=t}^{\infty} \frac{d_s}{(1+r)^s} \geq k_t \forall t \end{cases} \tag{12}$$

Sequence of forward looking state variables: daunting problem?

Contract with limited enforceability

Define V_t to be the value to the entrepreneur under some contract

- key insight (Spear and Srivastava 1987): use promised value to entrepreneur as a state variable
- recursive formulation

$$W(V) = \max_{\{k \geq 0, d \geq 0, V'\}} \pi(k) + \frac{1}{1+r} W(V') \tag{13}$$

$$s.t. \begin{cases} k \leq V \\ V' = (1+r)(V-d) \end{cases} \tag{14}$$

last constraint $\rightarrow$ entrepreneur Bellman equation

Optimal contract with limited enforceability

Remember first-best sets $k = k^*$

- limited enforceability prevents us from reaching that (immediately)

Optimal contract

- if $V \geq k^*$, set choose $k = k^*$ and d so that $V' = k^*$
- if $V < k^*$, choose $k = V$ and $d = 0$ (why?)

endogenous borrowing constraints

- the optimal contract features an initial loan $< I + k^*$

Optimal contract with limited enforceability

Simulation or how does the contract evolve

- start with $V_0 = U_0$ that $\implies$ entrepreneur accepts the contracts
- value for creditor is $U_I = W(V_0) - V_0$
- for next period, set $V_1 = (1+r)V_0$ and $k_0 = V_0$
- proceed this way until $k = k^*$
- then remain in optimal contract

how does firm value evolve?

- V grows at rate $r \implies$ capital k grows at rate r
- firm value grows until reaching optimal firm value $W^* = \sum_{t=1}^{\infty} \frac{\pi(k^*)}{(1+r)^t}$

Different implementations

- We have characterized the optimal allocation with and without limited enforceability
- the same allocation can be achieved with a different contract that would imply
  - different initial debt
  - different maturity structure
  - different evolution of equity
- dynamic contracts literature is rich and studies different variations of informational frictions, allocations and implementations