ECON 210: Section 10

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Plan for today

- Optimal control summary (deterministic + stochastic)
- Stochastic optimal control problem (Exam 2019)

Optimal Control Summary

Deterministic: Hamiltonian Approach

We are interested in solving

$$\max_{\{c(t)\}_{t\in[0,T]}} \int_0^T u(s(t), c(t), t)dt \tag{1}$$

$$\dot{s}(t) = g(s(t), c(t), t) \tag{2}$$

$$f(s(t), c(t), t) \geq 0 \tag{3}$$

$$s(0) \text{ given} \tag{4}$$

Optimal Control Summary

Deterministic: Hamiltonian Approach

Approach 1: follows the continuous-time analogue of lagrangian

$$H(s, c, \lambda, t) = u(s, c, t) + \lambda g(s, c, t) \tag{5}$$

Necessary conditions for optimality

i $\frac{\partial H(s^*, c^*, \lambda^*, t)}{\partial c} = 0$

ii $\frac{\partial H(s^*, c^*, \lambda^*, t)}{\partial s} = -\dot{\lambda}^*(t)$

iii $\frac{\partial H(s^*, c^*, \lambda^*, t)}{\partial \lambda} = \dot{s}^*$

iv boundary conditions $s^*(0) = s_0$

With concavity, we get sufficiency.

Optimal Control Summary

Deterministic: HJB Approach

Approach 2: follows the continuous-time analogue of the bellman equation

$$\begin{gathered} 0 = \max_{c(t)} \underbrace{u(s(t), c(t), t)}_{\text{“dividend-yield”}} + \underbrace{V_s(s(t), t)\dot{s}(t) + V_t(s(t), t)}_{\text{“capital-gains”}} \\ \text{s.t.} \\ \dot{s}(t) = g(s(t), c(t), t) \\ s(t) \text{ given} \end{gathered} \tag{6}$$

Optimal Control Summary

Finite and Infinite-Horizon

- In finite horizon, optimal policy or restrictions will imply $c(T)$ (e.g. "eat everything").
- In infinite horizon, we must add an analogue *boundary condition*, which is called the transversality condition

$$\lim_{T \to \infty} \lambda(T) s(T) = 0 \text{ or } \lim_{T \to \infty} e^{-\rho T} V(s_T) = 0 \tag{7}$$

Optimal Control Summary

Stochastic Case

- In continuous-time, to model "shocks" we use brownian motion (does not restrict to normality!)
- We proceed here with the HJB approach, but there is an analogue of the Hamiltonian approach (stochastic maximum principle)
- The stochastic analogue of the HJB follows the same principle as in the deterministic case and the discrete-time bellman equation

$$0 = \max_{c(t)} \underbrace{u(s(t), c(t), t)}_{\text{"dividend-yield"}} + \underbrace{E_t \left[dV(s,t) \right]}_{\text{expected capital gains}} \tag{8}$$

where we postulate the evolution of the state to follow the stochastic version of an ODE, a stochastic differential equation (SDE)

$$ds_t = \mu_{s,t} dt + \sigma_{s,t} dB_t \tag{9}$$

Optimal Control Summary

Stochastic Case

- We can use Ito's lemma to calculate $E_t[dV(s,t)]$
- Crucially, this transforms the problem of computing expectations into one of determining the derivatives of V
- The same equation can be written if the environment is subject to shocks of other nature in continuous-time, namely, jumps (shocks that arrive at some poisson rate).
- Similarly, we can extend the agent's control to affect the arrival or impact of these poisson shocks
- In short, very tractable, elegant and lots of flexibility!

Optimal Control Summary

Stochastic Case

A Particular, Common and Useful Case

When we have time-separable utility with exponential discounting (very common), the HJB equation reduces to

$$\underbrace{\rho V(s,t)}_{\text{“required total return on capital”}} = \max u(c) + V_s(s,t)\mu_s + V_t(s,t)\underbrace{\mu_t}_{=1} + \underbrace{\frac{1}{2}V_{ss}(s,t)\sigma_s^2}_{\text{Ito term}}$$

Great textbook reference for SDEs and control: Oksendal
At Stanford, MATH236 covers part of this book.

Optimal Control Summary

Stochastic Case - Further Topics

Further stochastic control problems not covered in this course:

- SDE's subject to jump shocks
- Marginal costs to exert control (instantaneous control)
- Fixed costs to exert control (impulse control)
- Optimal stopping

Excellent textbook and intuitive treatment of the last 3: Dixit "The Art of Smooth Pasting"